OLIMPIADA DE MATEMATICĂ

ETAPA LOCALĂ

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BAREM

CLASA A XI-A

1.)	**Din oficiu**	**1p**
	$x\cdot\begin{pmatrix}1&2&3&4&5&...&2016\\2&3&1&4&5&...&2016\end{pmatrix}\cdot x=e \;\Big\vert\; \cdot x^{-1}$ la stânga și la dreapta	**2p**
	$\begin{pmatrix}1&2&3&4&5&...&2016\\2&3&1&4&5&...&2016\end{pmatrix}=x^{-2}$	**1p**
	$x^2=\begin{pmatrix}1&2&3&4&5&...&2016\\3&1&2&4&5&...&2016\end{pmatrix}$	**1p**
	$\Rightarrow\begin{cases}x(x(1))=3\\x(x(2))=1\\x(x(3))=2\\x(x(4))=4\\...\\x(x(2016))=2016\end{cases}$	**1p**
	Caz 1. $x(1)=1\Rightarrow x(1)=3$ nu convine Caz 2. $x(1)=2\Rightarrow x(2)=3\Rightarrow x(3)=1\Rightarrow x(4)=4,\ldots$Găsim soluțiile: $x=\begin{pmatrix}1&2&3&4&5&...&2016\\2&3&1&4&5&...&2016\end{pmatrix}, x=\begin{pmatrix}1&2&3&4&5&...&2016\\2&3&1&5&4&...&2016\end{pmatrix}$ $x=\begin{pmatrix}1&2&3&4&5&6&7&...&2016\\2&3&1&6&5&4&7&...&2016\end{pmatrix},\ldots$care generează C_{2013}^2 soluții, $C_{2013}^2>2017$	**1p** **3p**
	Soluţie alternativă $\alpha=\begin{pmatrix}1&2&3&4&5&...&2016\\2&3&1&4&5&...&2016\end{pmatrix}=(1,2,3)\Rightarrow\alpha^3=e$	
	Găsim soluțiile: $x=(1,2,3)\cdot(4,5), x=(1,2,3)\cdot(4,6),..., x=(1,2,3)\cdot(4,2016)$ care generează C_{2013}^2 soluții, $C_{2013}^2>2017$	

2.)	**Din oficiu**	**1p**
	a) $A^n=6^{n-1}A$ inducție sau Cayley-Hamilton	**4p**
	b) Din $A^n=6^{n-1}A$ obținem: $A=\frac{1}{6^{n-1}}A^n$	**1p**
	Putem scrie: $A=\frac{6}{6^n}A^n=\frac{1}{6^n}A^n+\frac{2}{6^n}A^n+\frac{3}{6^n}A^n$ pentru $n=2017$ avem: $A=\frac{6}{6^{2017}}A^{2017}=\frac{1}{6^{2017}}A^{2017}+\frac{2}{6^{2017}}A^{2017}+\frac{3}{6^{2017}}A^{2017}$	**2p**
	Alegem $X=\frac{1}{6}A,\ Y=\frac{\sqrt[2017]{2}}{6}A,\ Z=\frac{\sqrt[2017]{3}}{6}A\quad ,X,Y,Z\in M_3(\mathbb{R}_+)$	**2p**

3.)	**Din oficiu**	**1p**
	a) $a_1=1=3^1-2^1$ $a_2=3\cdot a_1+2^1=3\cdot 1+2=3+2=(3-2)\cdot(3+2)=3^2-2^2$ $a_3=3\cdot a_2+2^2=3\cdot(3^2-2^2)+2^2=3^3-3\cdot 2^2+2^2==3^3-2\cdot 2^2=3^3-2^3$	**2p**
	Demonstrăm prin inducție matematică: $a_n=3^n-2^n,\ \forall n\in\mathbb{N}^*$ $a_{n+1}=3\cdot a_n+2^n=3\cdot(3^n-2^n)+2^n=3^{n+1}-3\cdot 2^n+2^n=3^{n+1}-2\cdot 2^n=3^{n+1}-2^{n+1},\ \forall n\in\mathbb{N}^*$	**2p**
	b) Presupunem că $\exists n\in\mathbb{N}^*$ astfel încât $\div a_{n-1},a_n,a_{n+1}\Leftrightarrow 2a_n=a_{n-1}+a_{n+1}$ obținem: $2(3^n-2^n)=3^{n-1}-2^{n-1}+3^{n+1}-2^{n+1}\Leftrightarrow 2^{n-1}=4\cdot 3^{n-1}\Leftrightarrow\left(\frac{2}{3}\right)^{n-1}=4$ Ecuația $\left(\frac{2}{3}\right)^{n-1}=4$ nu are soluții în $\mathbb{N}$ rezultă că nu există $n\in\mathbb{N}^*$ astfel încât $\div a_{n-1},a_n,a_{n+1}$	**2p**
	c) $\lim\limits_{n\to\infty}\frac{a_n}{n!}=\lim\limits_{n\to\infty}\frac{3^n-2^n}{n!}=\lim\limits_{n\to\infty}\frac{3^n}{n!}-\lim\limits_{n\to\infty}\frac{2^n}{n!}$ Cum $0<\frac{3^n}{n!}=\frac{3\cdot3\cdot3\cdot\ldots\cdot3\cdot3}{1\cdot2\cdot3\cdot4\cdot5\cdot\ldots\cdot n-1\cdot n}=\frac{3}{1}\cdot\frac{3}{2}\cdot\frac{3}{3}\cdot\frac{3}{4}\cdot\frac{3}{5}\cdot\ldots\cdot\frac{3}{n}<\frac{9}{2}\cdot\left(\frac{3}{4}\right)^{n-3}\to 0,\ n\to\infty$ avem $\lim\limits_{n\to\infty}\frac{3^n}{n!}=0$ și analog $\lim\limits_{n\to\infty}\frac{2^n}{n!}=0$ cee ace implică $\lim\limits_{n\to\infty}\frac{a_n}{n!}=\lim\limits_{n\to\infty}\frac{3^n-2^n}{n!}=0$	**1,5p**
	Pentru $k\geq 3$ obținem: $\lim\limits_{n\to\infty}\frac{a_n}{2^n+k^n}=\lim\limits_{n\to\infty}\frac{3^n-2^n}{2^n+k^n}=\lim\limits_{n\to\infty}\frac{3^n\left(1-\left(\frac{2}{3}\right)^n\right)}{k^n\left(\left(\frac{2}{k}\right)^n+1\right)}=\lim\limits_{n\to\infty}\left(\frac{3}{k}\right)^n\frac{\left(1-\left(\frac{2}{3}\right)^n\right)}{\left(\left(\frac{2}{k}\right)^n+1\right)}=\begin{cases}0, \text{ dacă } k>3\\ 1, \text{ dacă } k=3\end{cases}$ Pentru $k<3$, avem: $\lim\limits_{n\to\infty}\frac{a_n}{2^n+k^n}=\lim\limits_{n\to\infty}\frac{3^n-2^n}{2^n+k^n}=\lim\limits_{n\to\infty}\frac{3^n\left(1-\left(\frac{2}{3}\right)^n\right)}{2^n\left(1+\left(\frac{k}{2}\right)^n\right)}=\lim\limits_{n\to\infty}\left(\frac{3}{2}\right)^n\frac{\left(1-\left(\frac{2}{3}\right)^n\right)}{\left(1+\left(\frac{k}{2}\right)^n\right)}=\infty$	**1,5p**

4.)	**Din oficiu**	**1p**
	$\sum\limits_{k=1}^{2016}2016^{\frac{k-n}{n}}=2016^{\frac{1-n}{n}}+2016^{\frac{2-n}{n}}+\ldots+2016^{\frac{2016-n}{n}}$	**1p**
	$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\left(2016^{\frac{1-n}{n}}+2016^{\frac{2-n}{n}}+\ldots+2016^{\frac{2016-n}{n}}\right)^n\overset{1^\infty}{=}$	**1p**
	$=\lim\limits_{n\to\infty}\left(1+2016^{\frac{1-n}{n}}+2016^{\frac{2-n}{n}}+\ldots+2016^{\frac{2016-n}{n}}-1\right)^n=$ $=\lim\limits_{n\to\infty}\left(1+2016^{\frac{1-n}{n}}-\frac{1}{2016}+2016^{\frac{2-n}{n}}-\frac{1}{2016}+\ldots+2016^{\frac{2016-n}{n}}-\frac{1}{2016}\right)^n=$	**2p**
	$=\lim\limits_{n\to\infty}\left(1+\frac{2016^{\frac{1}{n}}-1}{2016}+\frac{2016^{\frac{2}{n}}-1}{2016}+\ldots+\frac{2016^{\frac{2016}{n}}-1}{2016}\right)^n=$	**2p**

	$=e^{\lim\limits_{n\to\infty} n\cdot\left(\frac{2016^{\frac{1}{n}}-1}{2016}+\frac{2016^{\frac{2}{n}}-1}{2016}+...+\frac{2016^{\frac{2016}{n}}-1}{2016}\right)}=e^{\lim\limits_{n\to\infty}\frac{1}{2016}\cdot\left(\frac{2016^{\frac{1}{n}}-1}{\frac{1}{n}}+\frac{2016^{\frac{2}{n}}-1}{\frac{1}{n}}+...+\frac{2016^{\frac{2016}{n}}-1}{\frac{1}{n}}\right)}=$ $=e^{\lim\limits_{n\to\infty}\frac{1}{2016}\cdot\left(\frac{2016^{\frac{1}{n}}-1}{\frac{1}{n}}+2\cdot\frac{2016^{\frac{2}{n}}-1}{\frac{2}{n}}+...+2016\cdot\frac{2016^{\frac{2016}{n}}-1}{\frac{2016}{n}}\right)}$; $\frac{a^{x_n}-1}{x_n}\to\ln a,\ x_n\to 0$	**1p**
	$=e^{\frac{1}{2016}\cdot(\ln 2016+2\cdot\ln 2016+...+2016\cdot\ln 2016)}=$ $=e^{\frac{1}{2016}\cdot(1+2+...+2016)\cdot\ln 2016}=e^{\frac{2017}{2}\cdot\ln 2016}=e^{\ln 2016^{\frac{2017}{2}}}=2016^{\frac{2017}{2}}=\sqrt{2016^{2017}}$	**2p**